Cutting Convex Polyhedra by Planes

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Polyhedral graphs

Steinitz’s theorem says that a graph $G$ is isomorphic to the 1-skeleton of a three-dimensional convex polyhedron if and only if $G$ is planar and 3-connected. By this reason 3-connected planar graphs are called polyhedral.
Cutting planar graphs by lines

Let $\pi$ be a drawing of a graph $G$ and $\ell$ be a line. We say that $\ell$ crosses an edge or a face of $\pi$ if $\ell$ intersects it at an inner point.

Denote the number of edges (resp. faces) of $\pi$ that $\ell$ crosses by $\bar{e}(\pi, \ell)$ (resp. $\bar{f}(\pi, \ell)$).

\[
\bar{e}(G) = \max_{\pi, \ell} \bar{e}(G, \pi)
\]

\[
\bar{f}(G) = \max_{\pi, \ell} \bar{f}(G, \pi)
\]

Denote the number of vertices of $\pi$ on $\ell$ by $\bar{v}(\pi, \ell)$.

\[
\bar{v}(G) = \max_{\pi, \ell} \bar{v}(\pi, \ell)
\]
Example

$$\bar{v}(\pi, \ell) = 1 \quad \bar{e}(\pi, \ell) = 2, \quad \bar{f}(\pi, \ell) = 3.$$
Some Theorems

For every triangulation $T$, $\bar{v}(T) \leq \bar{f}(T) = \bar{e}(T) \leq c(T^*)$.

(The circumference $c(G)$ of a graph $G$ is the length of a longest cycle in $G$ and $G^*$ is the dual of a polyhedral graph $G$).

If $G$ be a planar graph such that degree of each vertex of $G$ is at least $k$ then $\bar{e}(G) \geq (k/2 - 1)\bar{v}(G)$. In particular, $\bar{e}(G) \geq (1/2)\bar{v}(G)$ for each polyhedral graph $G$. 
Cutting convex polyhedra by planes

Let $G$ be a polyhedral graph, $\pi$ be a convex polyhedron whose 1-skeleton is isomorphic to $G$, and $\ell$ be a plane. Values $\overline{e}(G)$, $\overline{f}(G)$, and $\overline{v}(G)$ are defined similarly to $\overline{\bar{e}}(G)$, $\overline{\bar{f}}(G)$, and $\overline{\bar{v}}(G)$ from the preceding section.
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Let $G$ be a polyhedral graph, $\pi$ be a convex polyhedron whose 1-skeleton is isomorphic to $G$, and $\ell$ be a plane. Values $\bar{e} (G)$, $\bar{f} (G)$, and $\bar{v} (G)$ are defined similarly to $\bar{e}(G)$, $\bar{f}(G)$, and $\bar{v}(G)$ from the preceding section.

$$\bar{v} (\pi, \ell) = 2, \quad \bar{e} (\pi, \ell) = 3, \quad \bar{f} (\pi, \ell) = 3.$$
For every polyhedral graph $G$, $\bar{v}(G) \leq \bar{f}(G) = \bar{e}(G) \leq c(G^*)$.

The last inequality was used by Grünbaum to show that for some $G$ we have $\bar{e}(G) = O(n^\alpha)$ for some $\alpha < 1$.

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Relations between 2-d and 3-d cases

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Is $f(G) = \bar{f}(G)$?

Is there any relation between $\bar{v}(G)$ and $\bar{v}(G)$?
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There are polyhedral graphs $G$ on $n$ vertices with $\bar{v}(G) > (2/3)n - 2$ and $c(G) = O(n^{\log_3 2})$. 
Polyhedra with small planar sets of vertices

The shortness exponent of a class of graphs $\mathcal{G}$ is the limit inferior of quotients $\log c(G)/\log v(G)$ over all $G \in \mathcal{G}$. Let $\sigma$ denote the shortness exponent for the class of cubic polyhedral graphs. It is known that

$$0.753 < \sigma \leq \frac{\log 22}{\log 23} = 0.985\ldots$$
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For each \( \alpha > \sigma \) there is a sequence of triangulations \( G \) with \( \bar{\nu}(G) = O(n^\alpha) \).
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Drawing graphs on several planes

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Proof. Let $V(G) = \{v_1, \ldots, v_n\}$, $\chi : V(G) \to \{1, \ldots, \chi(G)\}$ be a coloring of $G$, and $x_1, \ldots, x_n$ be real numbers which are linearly independent over the field $\mathbb{Q}$. Then $d(v_i) = (x_i, \chi(v_i))$ is the required drawing.
Drawing graphs on several planes

$\pi(G)$ is equal to the smallest size $r$ of a partition $V(G) = V_1 \cup \ldots \cup V_r$ such that every $V_i$ induces a planar subgraph of $G$. Therefore,

$$\frac{1}{4} \chi(G) \leq \pi(G) \leq \chi(G)$$
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Let $\rho(G)$ denote the minimum number of planes in the space such that a graph $G$ can be drawn on these planes (that is, every edge lies on one of the planes).
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Compute $\rho(K_n)$. 